Numerical Integration

Trapezoidal rule

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

Simpson's $\frac{1}{3}$ rule

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots+y_{n-1})+2(y_2+y_4+y_6+\cdots+y_{n-2})\right]$$

Simpson's $\frac{3}{8}$ rule

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4)+2(y_3+y_6+y_9+\cdots)\right]$$

Note

		order	Error
1.	Trapezoidal rule	h^2	$E = -\frac{(b-a)}{12}h^2 y''(\bar{x})$
2.	Simpson's $\frac{1}{3}$ rule	h^4	$E = -\frac{(b-a)}{180}h^4 y^{iv}(\bar{x})$
3.	Simpson's $\frac{3}{8}$ rule	h^5	$E = -\cancel{(b-a)}\frac{3}{80}h^5 y^{iv}(\bar{x})$

Problem : 1

Evaluate $\int_{4}^{5.2} \log_e x \, dx$ by using (i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$ rule & (iii) Simpson's $\frac{3}{8}$ rule, given that

x	4	4.2	4.4	4.6	4.8	5.0	5.2
$\log_e x$	1.386	1.435	1.482	1.526	1.569	1.609	1.649

Solution:

Here $h = 0.2$

$y_0 = 1.386$ $\quad y_1 = 1.435$ $\quad y_2 = 1.482$ $\quad y_3 = 1.526$

$y_4 = 1.569$ $\quad y_5 = 1.609$ $\quad y_6 = 1.649$

(i) Trapezoidal rule.

$$\int_{4}^{5.2} \log_e x \, dx = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{(0.2)}{2}\left[(1.386+1.649) + 2\begin{pmatrix}1.435+1.482+1.526+\\1.569+1.609\end{pmatrix}\right]$$

$$= (0.1)(18.277)$$

$$= 1.8277$$

(ii) Simpson's $\frac{1}{3}$ rule.

$$\int_{4}^{5.2} \log_e x \, dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{0.2}{3}\left[(1.386+1.649) + 4(1.435+1.526+1.609) + 2(1.482+1.569)\right]$$

$$= 1.8278$$

(iii) Simpson's $\frac{3}{8}$ rule

$$\int_4^{5.2} \log_e x\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$= \frac{3(0.2)}{8}\left[(1.386+1.649)+3(1.435+1.482+1.569+1.609)+2(1.526)\right]$$

$$\int_4^{5.2} \log_e x\,dx = 1.8279$$

2) Evaluate $\int_0^{\pi} \sin x\,dx$ by dividing the interval into 8 strips using (i) Trapezoidal rule

(ii) Simpson's $\frac{1}{3}$ rule.

Solution:

$$h = \frac{b-a}{n} = \frac{\pi-0}{8} = \frac{\pi}{8}$$

Given $y = \sin x$

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{6\pi}{8}$	$\frac{7\pi}{8}$	π
$\sin x$	0 y_0	0.3827 y_1	0.7071 y_2	0.9239 y_3	1 y_4	0.9239 y_5	0.7071 y_6	0.3827 y_7	0 y_8

(i) Trapezoidal rule.

$$\int_0^{\pi} \sin x\,dx = \frac{h}{2}\left[(0+0)+2(0.3827+0.7071+0.9239+1+0.9239+0.7071+0.3827)\right]$$

$$= \frac{\frac{\pi}{8}}{2}\left[10.0548\right]$$

$$= \frac{\pi}{16}\left[10.0548\right]$$

$$\int_0^{\pi} \sin x\,dx = 1.97425$$

(ii) Simpson's $\frac{3}{8}$ rule

$$\int_0^{\pi} \sin x\,dx = \frac{3h}{8}\left[(y_0+y_8) + 4(y_1+y_3+y_5+y_7) + 2(y_2+y_4+y_6)\right]$$

$$= \frac{3\left(\frac{\pi}{4}\right)}{8}\left[(0+0)+4(0.3827+0.9239+0.9239+0.7071) + 2(0.7071+1+0.7071)\right]$$

$$= \frac{3\pi}{24}\left[10.4528 + 2(2.4142)\right]$$

$$\int_0^{\pi} \sin x\,dx = 2.0003$$

3) Find $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule. Hence obtain the approximate value of π in each case

Solution:

we divide the range (0,1) into six equal parts.

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}.$$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y=\frac{1}{1+x^2}$	1 y_0	0.9730 y_1	0.9 y_2	0.8 y_3	0.6923 y_4	0.5902 y_5	0.5 y_6

By Simpson's $\frac{1}{3}$ rule.

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{\frac{1}{6}}{3}\left[(1+0.5)+4(0.9730+0.8+0.5902)+ 2(0.9+0.6923)\right]$$

$$= 0.7854. \quad —①$$

Simpson's $\frac{3}{8}$ rule.

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$= \frac{3\left(\frac{1}{6}\right)}{8}\left[(1+0.5)+3(0.9730+0.8+0.6923+0.5902)+2(0.8)\right]$$

$$= \frac{1}{16}\left[1.5+3(3.1555)+1.6\right]$$

$$= 0.7854 \qquad \text{--- (2)}$$

By Actual Integration.

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \left[\tan^{-1}(1)-\tan^{-1}(0)\right]$$

$$= \frac{\pi}{4}$$

from (1) & (2), we have

$$\frac{\pi}{4} = 0.7854.$$

$$\pi = 4(0.7854)$$

$$\boxed{\pi = 3.1416}$$

4) Evaluate $\int_0^{1.2} e^{-x^2}\,dx$ using (i) Simpson's $\frac{1}{3}$ rule, (ii) Simpson's $\frac{3}{8}$ rule taking $h=0.2$

solution:

Let $y = e^{-x^2}$ & G.T $h=0.2$

x	0	0.2	0.4	0.6	0.8	1	1.2
$y=e^{-x^2}$	1	0.9608	0.8521	0.6977	0.5273	0.3679	0.2369

(i) By simpson $\frac{1}{3}$ rule

$$\int_0^{1.2} e^{-x^2}\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{0.2}{3}\left[(1+0.2369)+4(0.9608+0.6977+0.3679) + 2(0.8521+0.5273)\right]$$

$$= 0.8067$$

(ii) Simpson's $\frac{3}{8}$ rule

$$= \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$= \frac{3(0.2)}{8}\left[(1+0.2369)+3(0.9608+0.8521+0.5273+0.3679)+2(0.6977)\right]$$

$$= 0.8067$$

Homework.

1. Evaluate $\int_0^{\pi/2}\sqrt{\cos\theta}\,d\theta$ by using simpson's rule of integration with 7 ordinates. $(7=6+1)$.

~~Sol~~ Hint. take $n=6$. $h = \frac{\frac{\pi}{2}-0}{6} = \frac{\pi}{12}$

$$\int_0^{\pi/2}\sqrt{\cos\theta}\,d\theta = 1.1873. \text{ (simpson } \tfrac{1}{3} \text{ rule)}$$

2. Evaluate $\int_0^{\pi/2} e^{\sin x}\,dx$, correct to 4 decimal places, using simpson $3/8$ rule.

Hint $n=6$. $h=\frac{\pi}{12}$ Ans: 3.1051.

3. Evaluate $\int_0^{\pi} \frac{\sin x}{x} dx$, by dividing the range into six equal parts using Simpson's rule.

Hint:

$n = 6 \quad h = \frac{\pi - 0}{6} = \frac{\pi}{6}$.

By Simpson's 1/3 rule $\int_0^{\pi} \frac{\sin x}{x} dx = 1.8520$.

4. Evaluate $\int_0^4 e^x dx$, by Simpson's rule, given that $e^1 = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.60$ and compare it with the actual value.

Hint:

By Simpson's $\frac{1}{3}$ rule

$\int_0^4 e^x dx = 53.8733$.

∴ Actual value $\int_0^4 e^x dx = \left[e^x \right]_0^4 = \left[e^4 - e^0 \right]$

$= e^4 - 1$

$= 53.60$.

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